# YAU COLLEGE MATH CONTESTS ALL AROUND ALGEBRA 2018

**Problem 1**

Let $\zeta_n$ be a primitive $n$-th root of unity, where $n > 1$ is a positive integer. Show that the number

$$N := \prod_{1 \le i \le n, (i,n)=1} (1 - \zeta_n^i)$$

is $p$ if $n$ is a power of a prime $p$, and is 1 if $n$ is not a power of a prime (i.e., $n$ is divisible by at least two distinct primes).

**Problem 2**

Let $F$ be a field. Let $G = GL_2(F)$ and $B$ be the subgroup of $G$ consisting of all upper triangular matrices. Then $B \times B$ acts on $G$ by left and right multiplication, i.e.

$$B \times B \times G \longrightarrow G, \quad (b_1, b_2, a) = b_1 a b_2^{-1}.$$

Prove that there are exactly two orbits, and they can be represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Generalize this to $GL_n(F)$.

**Problem 3**

Let $M_n(\mathbb{Q})$ be the ring of all $n \times n$ matrices with coefficients in $\mathbb{Q}$ for a positive integer $n$. Describe all field extensions $K$ of $\mathbb{Q}$ such that there is an injective ring homomorphism $K \to M_n(\mathbb{Q})$. (Note: we take the convention that a ring homomorphism maps the additive identity and the multiplicative identity respectively to the additive identity and the multiplicative identity.)

**Remark**. This is a special case of classification of commutative sub-algebras in central simple algebras.

**Problem 4**

Fix positive integers $n > m > k$, and fix a $\mathbb{C}$-linear subspace $E \subset \mathbb{C}^n$ of dimension $k$. Let

$$X = \{\mathbb{C}\text{-linear subspace } V \subset \mathbb{C}^n | V \supset E,\ \dim V = m\}.$$

Does $X$ naturally have the structure of a compact manifold? If so, what is $\dim_{\mathbb{R}} X$? Does $X$ naturally have the structure of the coset space of a group?